On the representation of stresses at the surface of a solid or liquid exposed to an electric or magnetic field

1999 J. Phys.: Condens. Matter 11 3425

(http://iopscience.iop.org/0953-8984/11/17/301)

J. Phys.: Condens. Matter **11** (1999) 3425–3430. Printed in the UK PII: S0953-8984(99)97033-9

On the representation of stresses at the surface of a solid or liquid exposed to an electric or magnetic field

G Kloos

Technical Bureau for Optics and Electromechanics, Aldinger Strasse 4, D-70736 Fellbach, Germany

Received 27 August 1998, in final form 19 February 1999

Abstract. It is a striking feature of the macroscopic theory of forces in an electric or magnetic field that the corresponding stresses can be described by tensors of a completely analogous form while the corresponding force laws are quite different. It is shown that this feature stems from the fact that a more general mathematical equation that links a vector term to the corresponding tensor divergence contains the electromechanical and magnetomechanical force–stress relations under consideration as special cases. This relation is of some use in organizing the knowledge on electrostatic and magnetostatic stresses and in elucidating under which physical conditions a simple transition from the force law to the tensor expression is possible.

1. Introduction

It is often convenient to express the force density $\vec{f}$ acting on a solid or liquid using the stress tensor σ [1]:

$$\int_V \vec{f}\,\mathrm{d}V = \oiint_{\partial V} \sigma\,\mathrm{d}\vec{S} \tag{1}$$

where V is the volume and S is the corresponding closed surface. The force density and the stress tensor are then linked by what can be considered a generalization of Gauss's theorem:

$$\int_V f_i\,\mathrm{d}V = \int_V \partial_j \sigma_{ij}\,\mathrm{d}V = \oiint_{\partial V} \sigma_{ij}\,\mathrm{d}S_j \tag{2}$$

where ∂_j is used as shorthand for $\partial/\partial x_j$. In this article, the Einstein convention for summation is used, i.e., if an index occurs twice in a term, summation with respect to that index is implicitly understood. It follows from equation (2) that the force density can be interpreted as the tensor divergence of the stress tensor:

$$f_i = \partial_j \sigma_{ij}. \tag{3}$$

A well-known example is Hooke's law in anisotropic form [2]

$$Y_{ij} = c_{ijmn}\varepsilon_{mn} \tag{4}$$

where Y_{ij} is the tensor of mechanical stresses, c_{ijmn} is the tensor of elastic constants, and ε_{mn} is the strain tensor. The force law corresponding to equation (4) is

$$f_i = \partial_j Y_{ij}. \tag{5}$$

To study electromechanics or magnetomechanics, a generalization is necessary. The law of conservation of linear momentum that links continuum mechanics and Maxwell's theory of electromagnetism provides a basis for a macroscopic treatment of electromechanical [1] and magnetomechanical interaction.

Ignoring thermal stresses, the force density of a dielectric that is exposed to an electric field of sufficiently high amplitude can be expressed in the following form [3, 4]:

$$f_i = \partial_j Y_{ij} + \operatorname{div} \vec{D}\, E_i - \frac{1}{2} E_k E_l\, \partial_i \epsilon_{kl} - \frac{1}{2}[\operatorname{rot}(\vec{E} \times \vec{D})]_i + \partial_j \Phi_{ij}^{es} \tag{6}$$

where E_i and D_i are components of the electric field strength and the electric displacement vector, respectively, div $\vec{D}$ is the density of charge, ϵ_{ij} are the components of the dielectric tensor, and Φ_{ij}^{es} represents the components of the electrostrictive stress tensor. Equation (6) can be interpreted as an extension of Hooke's law to include electrical phenomena. The second term on the right-hand side of equation (6) corresponds to the Coulomb force. The third term vanishes inside a homogeneous dielectric medium and becomes important when inhomogeneities and discontinuities of the dielectric properties occur. It is the mathematical expression of the observation that the medium with the higher dielectric constant will tend to pull the less polarizable medium out of the electric field. A well-known application example is Quinke's method for measuring dielectric constants of liquids. The fourth term represents couples and is often considered to be of minor importance. It vanishes if $\vec{E}$ and $\vec{D}$ are parallel.

The tensor expression corresponding to equation (6) is

$$\sigma_{ij} = Y_{ij} + E_i D_j - \frac{1}{2}\delta_{ij} E_k D_k - \frac{1}{2}(E_i D_j - E_j D_i) + \Phi_{ij}^{es} \tag{7}$$

where the second and third terms on the right-hand side correspond to the form of the electrostatic stress tensor as it is frequently used in the technical literature. It should be stated that reference [5] contains an error. The last term appearing in equation (4) of reference [5] has not been taken into account in the tensor representation used in that article.

The force-density expression describing the magnetomechanical interaction, if thermal influences are discarded, reads

$$f_i = \partial_j Y_{ij} + [(\operatorname{rot} \vec{H}) \times \vec{B}]_i - \frac{1}{2} H_k H_l\, \partial_i \mu_{kl} - \frac{1}{2}[\operatorname{rot}(\vec{H} \times \vec{B})]_i + \partial_j \Phi_{ij}^{ms} \tag{8}$$

where H_i and B_i are the components of the vectors of the magnetic field strength and magnetic flux density, respectively, and μ_{ij} are the components of the tensor of magnetic susceptibility. The third and fourth tensors allow for an analogous interpretation as in the electric case. $[(\operatorname{rot} \vec{H}) \times \vec{B}]_i$ is the ith component of the vector product of the curl of the magnetic field strength vector and the magnetic flux-density vector, and represents the Lorentz force density if the derivative of the electric displacement vector with respect to time vanishes. Φ_{ij}^{ms} stands for the components of the magnetostrictive stress tensor.

The corresponding stress tensor is

$$\sigma_{ij} = Y_{ij} + H_i B_j - \frac{1}{2}\delta_{ij} H_k B_k - \frac{1}{2}(H_i B_j - H_j B_i) + \Phi_{ij}^{ms}. \tag{9}$$

The physical motivation for such transitions from an expression in terms of force densities to a tensor representation is that a tensor description of material properties fits well in the thermodynamic framework used to interpret and evaluate measurements from a macroscopic point of view [6–8].

In the literature, the second, third and fourth terms on the right-hand sides of equations (7) and (9) are designated as electrostatic or magnetostatic Maxwell stresses, respectively. Comparison of equations (6) and (8) shows that the underlying force laws differ fundamentally. It is surprising that the same tensor representation is obtained. To study this problem, I restrict the analysis to Maxwell stresses.

2. Method of analysis

To better understand what the two physical cases have in common and in which respects they differ, a mathematical consideration is undertaken with the aim of working out the essential features. The mathematical task consists in finding a tensor whose divergence corresponds to a given vector function. Vector notation rather than index notation is used in this part, because this should lead to the vector equations that will be used being written down in a clearer way.

Some steps will seem rather motiveless at first glance. In fact, the result has been found by trial and error and by comparison with results given in the literature [3–4, 9–11]. While I can present a consistent derivation of the stress tensor corresponding to the force laws studied here, I am not aware of a direct integration method that works for any given force expression.

To find a more general relation between a vector function and the corresponding tensor divergence, the following vector is considered:

$$\vec{f} = \vec{F}\,\mathrm{div}\,\vec{G} - \vec{G} \times \mathrm{rot}\,\vec{F} - \frac{1}{2}F_k F_l\,\vec{\nabla}a_{kl} - \frac{1}{2}\,\mathrm{rot}(\vec{F} \times \vec{G}) \tag{10}$$

where the two vectors $\vec{F}$ and $\vec{G}$ are interlinked by a symmetric matrix

$$G_i = a_{ij}F_j. \tag{11}$$

Theorem 1. *It will be shown that*

$$\vec{f} = \vec{F}\,\mathrm{div}\,\vec{G} - \vec{G} \times \mathrm{rot}\,\vec{F} - \frac{1}{2}F_k F_l\,\vec{\nabla}a_{kl} - \frac{1}{2}\,\mathrm{rot}(\vec{F} \times \vec{G}) = \vec{\mathrm{D}}\mathrm{iv}[S] + \vec{\mathrm{D}}\mathrm{iv}[W] = \vec{\mathrm{D}}\mathrm{iv}[T] \tag{12}$$

with

$$S_{ij} = F_i G_j - \frac{1}{2}\delta_{ij}F_k G_k \qquad W_{ij} = -\frac{1}{2}(F_i G_j - F_j G_i)$$
$$G_i = a_{ij}F_j \qquad a_{ij} = a_{ji}.$$

Proof. The last term appearing in equation (10) can easily be expressed as a tensor divergence if the following relation is taken into account:

$$-\frac{1}{2}\,\mathrm{rot}(\vec{F} \times \vec{G}) = \vec{\mathrm{D}}\mathrm{iv}\,W \qquad \text{where } W_{ij} = -\frac{1}{2}(F_i G_j - F_j G_i). \tag{13}$$

□

The symbol $\vec{\mathrm{D}}\mathrm{iv}$ is used to indicate the difference between the tensor divergence and the usual divergence div. W is an antisymmetric tensor. It follows from equation (13) that

$$\vec{f} = \vec{F}\,\mathrm{div}\,\vec{G} - \vec{G} \times \mathrm{rot}\,\vec{F} - \frac{1}{2}F_k F_l\,\vec{\nabla}a_{kl} + \vec{\mathrm{D}}\mathrm{iv}\,W. \tag{14}$$

To express the other three terms appearing in equation (14) as a tensor divergence, it is convenient to use the following formula [12]:

$$\vec{G} \times \mathrm{rot}\,\vec{F} = \vec{\nabla}(\vec{G}\cdot\vec{F}) - (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{F}\cdot\vec{\nabla})\vec{G} - \vec{F} \times \mathrm{rot}\,\vec{G}. \tag{15}$$

Combining equations (14) and (15), one has

$$\vec{f} = \vec{F}\,\mathrm{div}\,\vec{G} - \vec{\nabla}(\vec{G}\cdot\vec{F}) + (\vec{G}\cdot\vec{\nabla})\vec{F} + (\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F} \times \mathrm{rot}\,\vec{G} - \frac{1}{2}F_k F_l\,\vec{\nabla}a_{kl} + \vec{\mathrm{D}}\mathrm{iv}[W]. \tag{16}$$

At this point, a vector–tensor relation is employed that can easily be verified by differentiation:

$$\vec{F}(\mathrm{div}\,\vec{G}) + (\vec{G}\cdot\vec{\nabla})\vec{F} - \frac{1}{2}\vec{\nabla}(\vec{F}\cdot\vec{G}) = \vec{\mathrm{D}}\mathrm{iv}[S] \qquad \text{with } S_{ij} = F_i G_j - \frac{1}{2}\delta_{ij}G_k F_k. \tag{17}$$

Using this relation, the vector function takes the following form:

$$\vec{f} = \vec{\mathrm{Div}}[S] - \frac{1}{2}\vec{\nabla}(\vec{F}\cdot\vec{G}) + (\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F}\times \mathrm{rot}\,\vec{G} - \frac{1}{2}F_k F_l\,\vec{\nabla}a_{kl} + \vec{\mathrm{Div}}[W]. \tag{18}$$

In a further step, an expression is used that can be verified by componentwise comparison taking equation (11) into account:

$$(\vec{F}\cdot\vec{\nabla})\vec{G} + \vec{F}\times \mathrm{rot}\,\vec{G} = a_{ij}F_i\,\vec{\nabla}F_j + F_k F_l\,\vec{\nabla}a_{kl}. \tag{19}$$

Equation (18) then reads

$$\vec{f} = \vec{\mathrm{Div}}[S] - \frac{1}{2}\vec{\nabla}(\vec{F}\cdot\vec{G}) + a_{kl}F_k\,\vec{\nabla}F_l + \frac{1}{2}F_k F_l\,\vec{\nabla}a_{kl} + \vec{\mathrm{Div}}[W]. \tag{20}$$

After forming the gradient $\vec{\nabla}(F_k G_k) = F_k a_{km}\,\vec{\nabla}F_m + F_k F_m\,\vec{\nabla}a_{km} + a_{km}F_m\,\vec{\nabla}F_k$, one has

$$\vec{f} = \vec{\mathrm{Div}}[S] + \frac{1}{2}a_{km}F_k\,\vec{\nabla}F_m - \frac{1}{2}G_k\,\vec{\nabla}F_k + \vec{\mathrm{Div}}[W]. \tag{21}$$

Assuming that $a_{ij} = a_{ji}$ holds, the two terms in the middle cancel out and equation (12) is obtained as the final result.

3. Application to physical force laws

With equation (12), a mathematical expression has been found that contains the electromechanical and magnetomechanical problems presented in the introduction as special cases.

A first conclusion that holds for both physical situations is that the symmetry of the tensor a_{ij}, which represents either the dielectric tensor ϵ_{ij} or the tensor of magnetic susceptibilities μ_{ij}, was important in the derivation.

Substituting $\vec{E}$ for $\vec{F}$, $\vec{D}$ for $\vec{G}$, and ϵ_{ij} for a_{ij}, equation (21) can be compared with equations (6) and (7). It can be concluded that the force densities are identical if rot $\vec{E}$ vanishes. This corresponds to a physical situation in which the right-hand side of the Maxwell equation rot $\vec{E} = -\partial_t\vec{B}$ is equal to zero.

Replacing $\vec{F}$ by $\vec{H}$, $\vec{G}$ by $\vec{B}$ and a_{ij} by μ_{ij} and making a comparison with equations (8) and (9), one finds that the magnetomechanical phenomena considered here are also contained as a special case under the condition that div $\vec{B} = \vec{0}$ holds, which is equivalent to a Maxwell equation.

For a physical interpretation and further analysis, it might be useful to note that the Maxwell stress tensor T given by equation (12) can be decomposed into a symmetric part and a 'hydrostatic' part, if the standard decomposition into a symmetric tensor and an antisymmetric tensor is applied to the tensor represented by $F_i G_j$:

$$T_{ij} = F_i G_j - \frac{1}{2}\delta_{ij}F_k G_k - \frac{1}{2}(F_i G_j - F_j G_i) = \frac{1}{2}(F_i G_j + F_j G_i) - \frac{1}{2}\delta_{ij}F_k G_k. \tag{22}$$

4. Numerical examples

To predict the stresses at the surface of a given material quantitatively, the dielectric or magnetic properties of the material and of the material that has an interface with it have to be taken into account. This can be made clear by working through an example.

The starting point for calculating the stress at a surface induced by a field F_i is to consider the superposition of the Maxwell stress tensors of the two media, which are labelled I and II here:

$$U_{ij} = T^{\mathrm{II}}_{ij} - T^{\mathrm{I}}_{ij}. \tag{23}$$

It is assumed that the field strength F_i^{II} in medium II is known and that it only has a component in the z-direction. The induced stress in the z-direction at a surface with its normal vector pointing in the z-direction will now be calculated.

First, the Maxwell stress tensors are considered separately. Using $F_1^{\mathrm{II}} = 0$ and $F_2^{\mathrm{II}} = 0$, it follows from equations (22) and (11) that

$$T_{33}^{\mathrm{II}} = \frac{1}{2} a_{33}^{\mathrm{II}} (F_3^{\mathrm{II}})^2. \tag{24}$$

Combining of equations (22) and (11) applied to T_{33}^{I} with equations (23) and (24) gives

$$U_{33} = \frac{1}{2} a_{33}^{\mathrm{II}} (F_3^{\mathrm{II}})^2 + \frac{1}{2}(F_1^{\mathrm{I}} G_1^{\mathrm{I}} + F_2^{\mathrm{I}} G_2^{\mathrm{I}}) - \frac{1}{2}(F_3^{\mathrm{I}} a_{31}^{\mathrm{I}} F_1^{\mathrm{I}} + F_3^{\mathrm{I}} a_{32}^{\mathrm{I}} F_2^{\mathrm{I}} + F_3^{\mathrm{I}} a_{33}^{\mathrm{I}} F_3^{\mathrm{I}}). \tag{25}$$

To find an expression in terms of the known field strength F_i^{II}, the continuity conditions that follow from the Maxwell equations have to be applied. The continuity condition for the transverse components of the field strength allows one to conclude that

$$F_1^{\mathrm{I}} = 0 \tag{26}$$

and

$$F_2^{\mathrm{I}} = 0 \tag{27}$$

and to simplify equation (25) accordingly:

$$U_{33} = \frac{1}{2} a_{33}^{\mathrm{II}} (F_3^{\mathrm{II}})^2 - \frac{1}{2} a_{33}^{\mathrm{I}} (F_3^{\mathrm{I}})^2. \tag{28}$$

The continuity of the normal component of the corresponding flux density gives another equation:

$$G_3^{\mathrm{I}} = G_3^{\mathrm{II}}. \tag{29}$$

Using equations (26), (27) this can be written as follows:

$$a_{33}^{\mathrm{I}} F_3^{\mathrm{I}} = a_{33}^{\mathrm{II}} F_3^{\mathrm{II}}. \tag{30}$$

Combining equations (28) and (30), the final expression for the normal stress acting at the boundary of the media I and II is obtained:

$$U_{33} = \frac{1}{2} a_{33}^{\mathrm{II}} \left(1 - \frac{a_{33}^{\mathrm{II}}}{a_{33}^{\mathrm{I}}}\right) (F_3^{\mathrm{II}})^2. \tag{31}$$

Formula (31) can be applied to stresses at the surface of a solid material exposed to air or in contact with a liquid in a homogeneous field. It can also be applied to predict the stresses at the boundary of two immiscible fluids or at the surface of a liquid covered with a gas, if the applied field can be assumed to be homogeneous to a good approximation. It should be noted that the consideration of electroded surfaces, which are often used in electromechanics, demands a modification of the analysis.

Finally, some numerical examples will be given.

In the magnetic case, equation (31) can be written as

$$U_{33} = \frac{1}{2} \mu_{33}^{\mathrm{II}} \left(1 - \frac{\mu_{33}^{\mathrm{II}}}{\mu_{33}^{\mathrm{I}}}\right) (H_3^{\mathrm{II}})^3.$$

Taking the values $\chi^r = -12.9 \times 10^{-6}$ for LiF and $\chi^r = -14.0 \times 10^{-6}$ for NaCl from the literature [13], arbitrarily setting $H_3^{\mathrm{II}} = 0.636 \times 10^6$ N V^{-1} s^{-1}, and using $\mu^r = 1 + \chi^r$ and

$\mu = \mu_0\mu^r$ with $\mu_0 = 1.257 \times 10^{-6}$ V s A^{-1} m^{-1}, and $\mu^{r,\text{air}} = 1.000\,0004$, the following values are obtained for the normal stress at a surface exposed to air:

$$\text{LiF: } U_{33} = 3 \text{ N m}^{-2}$$
$$\text{NaCl: } U_{33} = 4 \text{ N m}^{-2}.$$

The same crystals can also serve as examples for the dielectric case. The low-frequency dielectric constants of LiF and NaCl are 9.03 and 5.92 [14], respectively. Considering an applied electric field of 10^6 V m^{-1} and using $\epsilon = \epsilon_0\epsilon^r$ with $\epsilon_0 = 8.8542 \times 10^{-12}$ A s V^{-1} m^{-1} and

$$U_{33} = \frac{1}{2}\epsilon_0\left(1 - \frac{1}{\epsilon^r}\right)(E_3^{\text{II}})^2$$

for the stress at the surface of the dielectric crystal covered by air, one has

$$\text{LiF: } U_{33} = 4 \text{ N m}^{-2}$$
$$\text{NaCl: } U_{33} = 4 \text{ N m}^{-2}.$$

5. Conclusions

Equation (12) is helpful for organizing a systematic representation of electromechanical and magnetomechanical forces acting on solids or liquids. It gives some insight into under which physical conditions a simple transition from the force law to the tensor representation or vice versa is possible. The considerations presented here are valid for any material symmetry, but are restricted to 'linear materials', i.e. higher-order terms that might be necessary to describe ferroelectric and ferromagnetic materials are not included. The analogy on the 'tensor level' can be exploited as a tool to translate results found in electromechanics into magnetomechanical results, for example.

References

[1] Tiersten H F 1969 *Linear Piezoelectric Plate Vibrations* (New York: Plenum)
[2] Nye J F 1964 *Physical Properties of Crystals* (Oxford: Oxford University Press)
[3] Durand E 1966 *Électrostatique* vol 3 (Paris: Masson)
[4] Robinson F N H 1973 *Macroscopic Electromagnetism* (Oxford: Pergamon)
[5] Kloos G 1995 The dependence of electrostatic stresses at the surface of a dielectric on its orientation in an electric field *J. Phys. D: Appl. Phys.* **28** 2424–9
[6] Kloos G 1995 The correction of interferometric measurements of quadratic electrostriction for cross effects *J. Phys. D: Appl. Phys.* **28** 939–44
[7] Kloos G 1997 On the thermodynamic framework of the description of reversible magnetomechanical and magnetocaloric phenomena *J. Magn. Magn. Mater.* **172** 247–53
[8] Kloos G 1998 Magnetostatic Maxwell stresses and magnetostriction *Electr. Eng.* **81** 77–80
[9] Juretschke H J 1977 Simple derivation of the Maxwell stress tensor and electrostriction effects in crystals *Am. J. Phys.* **45** 277
[10] Stumpf H and Schuler W 1981 *Elektrodynamik* (Wiesbaden: Vieweg)
[11] Brunk G 1972 Über die Äquivalenz von Lastsystemen und die mechanischen Wirkungen elektromagnetischer Felder *Dissertation* Technical University of Berlin
[12] Grosche G, Ziegler V and Ziegler D (ed) 1989 *Bronstein/Semendjajew: Taschenbuch der Mathematik* (Moscow: Nauka and Leipzig: Teubner) p 566
[13] Haussühl S 1983 *Kristallphysik* (Weinheim: Physik) p 417
[14] Lowndes R P and Martin D H 1970 Dielectric constants of ionic crystals and their variations with temperature and pressure *Proc. R. Soc.* A **316** 351–75